

# Numerical analysis of quasinormal modes in nearly extremal Schwarzschild-de Sitter spacetimes

Shijun Yoshida\*

*Centro Multidisciplinar de Astrofísica - CENTRA, Departamento de Física,  
Instituto Superior Técnico, Av. Rovisco Pais 1, 1049-001 Lisboa, Portugal*

Toshifumi Futamase†

*Astronomical Institute, Graduate School of Science, Tohoku University, Sendai 980-8578, Japan*

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We calculate high-order quasinormal modes with large imaginary frequencies for electromagnetic and gravitational perturbations in nearly extremal Schwarzschild-de Sitter spacetimes. Our results show that for low-order quasinormal modes, the analytical approximation formula in the extremal limit derived by Cardoso and Lemos is a quite good approximation for the quasinormal frequencies as long as the model parameter  $r_1\kappa_1$  is small enough, where  $r_1$  and  $\kappa_1$  are the black hole horizon radius and the surface gravity, respectively. For high-order quasinormal modes, to which corresponds quasinormal frequencies with large imaginary parts, on the other hand, this formula becomes inaccurate even for small values of  $r_1\kappa_1$ . We also find that the real parts of the quasinormal frequencies have oscillating behaviors in the limit of highly damped modes, which are similar to those observed in the case of a Reissner-Nordström black hole. The amplitude of oscillating  $\text{Re}(\omega)$  as a function of  $\text{Im}(\omega)$  approaches a non-zero constant value for gravitational perturbations and zero for electromagnetic perturbations in the limit of highly damped modes, where  $\omega$  denotes the quasinormal frequency. This means that for gravitational perturbations, the real part of quasinormal modes of the nearly extremal Schwarzschild-de Sitter spacetime appears not to approach any constant value in the limit of highly damped modes. On the other hand, for electromagnetic perturbations, the real part of frequency seems to go to zero in the limit.

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## I. INTRODUCTION

The quasinormal modes (QNMs) of spacetimes containing black holes have been studied since the pioneering work of Vishveshwara [1], who first observed quasinormal ringing of a Schwarzschild spacetime in his numerical calculations. The main motivation to study the QNM of a black hole is twofold: One is to answer the question of whether the spacetime is stable or not, and the other to know what kind of oscillations will be excited in the spacetime when some perturbations are given. In an astrophysical point of view, the latter is quite important from the observational point of view because we could determine fundamental parameters of a black hole, such as the mass or the angular momentum, through the information of the QNMs. Thus, a large number of studies on the QNMs of spacetimes containing black holes have been done (for review, see, e.g., [2, 3]).

There is another interesting aspect of the QNMs in a black hole which is related to the quantum theory of gravity. Bekenstein and Mukhanov discussed the relationship between the fundamental area unit in the quantum theory of gravity and a Bohr transition frequency, applying Bohr's correspondence principle with a hydro-

gen atom to the quantum theory of a black hole [4, 5]. For a Schwarzschild black hole, they then derived the Bohr transition frequency  $\omega$ , given by

$$\omega = \ln k / 8\pi M, \quad (1)$$

where  $k = 2, 3, 4, \dots$ , and  $M$  stands for the mass of a black hole. Furthermore they predicted the value of  $k$  to be  $k = 2$ , and suggested that this frequency should be equal to classical oscillation frequencies of the black hole. A few years ago, Hod however noticed that if  $k = 3$ , the frequency given by formula (1) is in quite good agreement with the asymptotic frequency of the QNM of a Schwarzschild black hole in the limit of highly damped modes, and proposed to apply Bohr's correspondence principle in order to determine the value of the fundamental area unit in the quantum theory of gravity, namely the value of  $k$  [6] (see, also [7]). Since Hod's proposal, the QNMs with large imaginary frequencies of spacetimes including black holes have been attracted much attention, and a lot of papers related to this subject have appeared in order to see whether Hod's conjecture is applicable not only for a Schwarzschild black hole but also for other black hole spacetimes. For example, Motl analytically obtained an asymptotic constant value of the QNM frequencies of a Schwarzschild black hole [8], which had been obtained numerically by Nollert [9]. Motl and Neitzke [10], Berti and Kokkotas [11], Neitzke [12], and Andersson and Howls [13] studied the asymptotic behaviors of the QNMs in a Reissner-Nordström black hole. Berti et al. [14], and Hod [15] discussed the QNMs of a

\*Electronic address: yoshida@fisica.ist.utl.pt

†Electronic address: tof@astr.tohoku.ac.jp

Kerr black hole in the limit of highly damped modes. As for a Schwarzschild-de Sitter (SdS) black hole, Cardoso and Lemos [16] and Maassen van den Brink [17] analytically obtained asymptotic form of the QNM frequencies in almost the extremal limit, in which the cosmological horizon becomes very close to the black hole horizon (for the case of a Schwarzschild black hole in an anti-de Sitter spacetime, see, e.g., [18, 19]).

A large number of papers related to Hod's conjecture, which have recently appeared, suggest that Hod's conjecture is not universal, at least as it stands, even though it is applicable for Schwarzschild black holes in four and higher dimensions (for the higher dimensional case, see, e.g., [10, 12, 27, 28]). For example, a real part of the quasinormal frequencies in a Reissner-Nordström black hole appears not to go to any constant value in the limit of highly damped mode, but shows some periodic behaviors as the imaginary part is increased [10, 11, 12, 13]. This means that Hod's conjecture is not applicable in the Reissner-Nordström case because a real part of the quasinormal frequencies does not have a limit as the imaginary part goes to infinity. In such a situation, it is necessary to explore a problem whether or not there is another black hole spacetime in which Hod's conjecture is applicable. The purpose of this paper is to improve our understanding of this problem and we are concerned here with the QNMs of SdS spacetimes. SdS spacetime has no spatial infinity but has cosmological horizon, and if Hod's conjecture is applicable, it is interesting to see whether Hod's conjecture depends only on the black hole horizon, but not the cosmological horizon. Furthermore, recent observations show that the universe does have a non-zero positive cosmological constant. Therefore, SdS spacetimes are considered to be a good simple model of a black hole in the universe.

In this study, in particular, we calculate numerically the QNMs of nearly extremal SdS spacetimes for reason we describe below. We therefore assume the surface gravity  $\kappa_1$  at the black hole horizon to be  $\kappa_1 \leq 10^{-2} r_1^{-1}$ , where  $r_1$  stands for the coordinate radius of the black hole horizon. Note that the extremal limit of the SdS space corresponds to the limit of  $\kappa_1 \rightarrow 0$ . In a nearly extremal SdS black hole, as mentioned before, an analytical formula for quasinormal frequencies can be derived [16, 17]. One of the aims of this paper is to examine whether this analytical formula is correct in the limit of highly damped modes.

The paper is organized as follows. In §2 we present the basic equations for obtaining QNMs in the SdS spacetime using Leaver's continued fraction technique [20], which was extended to the case of the SdS spacetime by Moss and Norman [21]. Numerical results are given in §3, and §4 is devoted for conclusion.

## II. METHOD OF SOLUTIONS

In order to examine the QNMs of the SdS spacetime, we make use of the same formalism as that derived by Moss and Norman [21], who obtained low-order quasinormal frequencies of the SdS spacetime for a wide range of the model parameter. The line element of the SdS spacetime is given by

$$ds^2 = -\frac{\Delta}{r^2} dt^2 + \frac{r^2}{\Delta} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (2)$$

where

$$\Delta = r^2 - 2Mr - \frac{1}{3}\Lambda r^4. \quad (3)$$

Here,  $M$  and  $\Lambda$  stand for the mass of the black hole and the cosmological constant, respectively. If a non-negative cosmological constant is assumed, namely de Sitter spacetime, there are two horizons, whose radial coordinates are given as positive solutions of  $r^{-1}\Delta = r - 2M - \frac{1}{3}\Lambda r^3 = 0$ , one is the black hole horizon  $r = r_1$  and the other the cosmological horizon  $r = r_2$ , where  $r_2 > r_1 > 0$ . Note that the equation  $r - 2M - \frac{1}{3}\Lambda r^3 = 0$  has one negative solution  $r = r_3 < 0$  for the SdS space. One of the important model parameters of the SdS spacetime is the surface gravity  $\kappa_1$ , defined by

$$\kappa_1 = \lim_{r \rightarrow r_1} \frac{1}{2r^2} \frac{d\Delta}{dr}. \quad (4)$$

In terms of the non-dimensional surface gravity  $r_1\kappa_1$ , the mass and the cosmological constant can be written as

$$M = \frac{1}{3}r_1(r_1\kappa_1 + 1), \quad \Lambda = r_1^{-2}(1 - 2r_1\kappa_1), \quad (5)$$

which shows that  $0 < r_1\kappa_1 < 1/2$  for the SdS spacetime. In this study, we employ the non-dimensional parameter  $r_1\kappa_1$  to specify the SdS spacetime.

By virtue of the symmetry properties of the SdS spacetime, the master equations for the scalar ( $s = 0$ ), electromagnetic ( $s = 1$ ) and gravitational perturbations ( $s = 2$ ) can be cast into a wave equation of the simple form, given by [22, 23, 24]

$$\frac{d^2\phi(r)}{dr_*^2} + [\omega^2 - V(r)]\phi(r) = 0, \quad (6)$$

where  $r_*$  denotes the tortoise coordinate, defined by  $dr_* = r^2\Delta^{-1}dr$ , and  $\omega$  is the oscillation frequency of the perturbations. Depending on the type of perturbations, here, the effective potential is explicitly given by

$$\frac{r^4 V}{\Delta} = \begin{cases} l(l+1) + \frac{2M}{r} - \frac{2\Lambda r^2}{3} & \text{for } s = 0 \\ l(l+1) & \text{for } s = 1 \\ l(l+1) - \frac{6M}{r} & \text{for } s = 2, \end{cases} \quad (7)$$

where  $l$  means the angular quantum number of perturbations. Here, only the axial parity perturbations have

been considered for the gravitational case because quasinormal frequencies of the polar parity perturbations are the same as those of the axial parity perturbations (for the proof, see Appendix).

The QNMs of the SdS spacetime are characterized by the boundary conditions of incoming waves at the black hole horizon and outgoing waves at the cosmological horizon, given by

$$\phi(r) \rightarrow \begin{cases} e^{-i\omega r_*} & \text{as } r_* \rightarrow \infty \\ e^{i\omega r_*} & \text{as } r_* \rightarrow -\infty, \end{cases} \quad (8)$$

where the time dependence of perturbations has been assumed to be  $e^{i\omega t}$ . In general, it is impossible to adapt the boundary condition (8) in a straightforward numerical integration to obtain quasinormal frequencies. Some special technique is therefore required for computations of QNMs. In the present investigation, we employ a standard technique devised by Leaver, namely the continued fraction method [20, 21].

To apply the continued fraction method to the SdS spacetime, it is convenient to introduce a new independent variable, defined by  $x = r^{-1}$ . With this new variable  $x$ , the asymptotic form of the perturbations as  $r_* \rightarrow \infty$  can be rewritten as

$$e^{-i\omega r_*} = (x - x_1)^{-\rho_1} (x - x_2)^{-\rho_2} (x - x_3)^{-\rho_3}, \quad (9)$$

where  $x_i = r_i^{-1}$  and  $\rho_i = i\omega/(2\kappa_i)$  for  $i = 1, 2, 3$ , where

$$\begin{aligned} \kappa_1 &= M(x_1 - x_2)(x_1 - x_3), \\ \kappa_2 &= M(x_2 - x_1)(x_2 - x_3), \\ \kappa_3 &= M(x_3 - x_1)(x_3 - x_2). \end{aligned} \quad (10)$$

The perturbation function  $\phi$  is expanded around the black hole horizon as

$$\phi = (x - x_1)^{\rho_1} (x - x_2)^{-\rho_2} (x - x_3)^{-\rho_3} \sum_{n=0}^{\infty} a_n \left( \frac{x - x_1}{x_2 - x_1} \right)^n, \quad (11)$$

where  $a_0 = 1$  and  $a_n$ 's for  $n \geq 1$  are determined by the three term recurrence relation, given by

$$\alpha_n a_{n+1} + \beta_n a_n + \gamma_n a_{n-1} = 0, \quad (12)$$

where

$$\alpha_n = 2M(x_1 - x_3)\{n^2 + 2(\rho_1 + 1)n + 2\rho_1 + 1\}, \quad (13)$$

$$\begin{aligned} \beta_n &= -2M(2x_1 - x_2 - x_3) \times \\ &\quad \{n^2 + (4\rho_1 + 1)n + 4\rho_1^2 + 2\rho_1\} - l(l + 1) \\ &\quad + 2Mx_1(s^2 - 1), \end{aligned} \quad (14)$$

$$\gamma_n = 2M(x_1 - x_2)(n^2 + 4\rho_1 n + 4\rho_1^2 - s^2). \quad (15)$$

Here,  $s = 2$  for the gravitational perturbations and  $s = 1$  for the electromagnetic perturbations. Note that in the present study, we do not consider the scalar perturbations because Leaver's method cannot be directly applied to the scalar case. Comparing the expanded eigenfunction

(11) with equation (9), we can see that the eigenfunction (11) satisfies the QNM boundary condition (8) if the power series in equation (11) converges for  $x_2 \leq x \leq x_1$ . This convergence condition is equivalent to the condition written in terms of continued fractions [25], which is given by

$$0 = \beta_0 - \frac{\alpha_0 \gamma_1}{\beta_1 -} \frac{\alpha_1 \gamma_2}{\beta_2 -} \dots \quad (16)$$

Therefore, we solve this algebraic equation to obtain the quasinormal frequency.

### III. NUMERICAL RESULTS

In this study, we are concerned with asymptotic behaviors of high-order QNMs, namely quasinormal frequencies with large imaginary parts, in almost the extremal limit, in which two horizons are quite near. We therefore consider only the case of  $r_1 \kappa_1 \leq 10^{-2}$ . For these small values of  $r_1 \kappa_1$ , the continued fractions (16) converge very quickly and Leaver's method works quite well even when quasinormal frequencies have quite large imaginary parts. Note that for moderate values of  $r_1 \kappa_1$ , however, the convergence of the continued fractions gets worse and Leaver's method is applicable only for modes with smaller imaginary frequencies. For those cases, thus, some other techniques such as Nollert's method [9] or a phase integral method [26] should be used to obtain high-order QNMs.

In order to check our numerical code, we have calculated fundamental frequencies of the QNMs for several values of  $r_1 \kappa_1$  and have fitted the mode frequencies as a function of  $r_1 \kappa_1$  with the polynomials defined by

$$\begin{aligned} \text{Re}(\omega r_1) &= r_1 \kappa_1 b_0 (1 - b_1 r_1 \kappa_1), \\ \text{Im}(\omega r_1) &= r_1 \kappa_1 c_0 (1 - c_1 r_1 \kappa_1). \end{aligned} \quad (17)$$

Recently, Cardoso and Lemos [16] and Maasse van den Brink [17] analytically obtained the expansion coefficients, which are given by

$$b_0 = \begin{cases} \sqrt{l(l+1) - \frac{1}{4}} & \text{for } s = 0, 1 \\ \sqrt{l(l+1) - \frac{9}{4}} & \text{for } s = 2, \end{cases} \quad (18)$$

$$c_0 = n + \frac{1}{2}, \quad b_1 = c_1 = \frac{2}{3}, \quad (19)$$

where  $n$  is the mode number. It is found that numerically obtained coefficients are in good agreement with coefficients given by (18) and (19). Note that our numerical results for the mode frequencies are consistent with those obtained by Moss and Norman [21], who studied low-order QNMs of the gravitational perturbations for full range of the parameter  $r_1 \kappa_1$ .

First, let us discuss properties of the quasinormal frequencies for the low-order modes. In Figure 1, we show

the real parts of the frequencies for the low-order QNMs versus the imaginary parts of the frequencies for the gravitational perturbations with  $l = 2$ . In this figure, non-dimensional frequencies  $\text{Re}(\omega/\kappa_1)$  have been plotted as a function of  $\text{Im}(\omega/\kappa_1)$  for  $r_1\kappa_1 = 10^{-3}$  and  $r_1\kappa_1 = 5 \times 10^{-3}$ , and the dashed curve indicates the approximate frequency for  $r_1\kappa_1 \rightarrow 0$ , derived by Cardoso and Lemos [16] (see, also [17]). For low-order modes, Figure 1 illustrates how the frequencies of the QNMs behaves when the mode number and/or the value of  $r_1\kappa_1$  is altered. Basic properties of the low-order QNMs are summarized as follows: For the modes associated with a small mode number, the real parts of the frequencies decrease with the increase of the imaginary parts of the frequency, even though the real parts of the frequency are constant in the approximation formula (17)-(19). In other words, the analytical approximation formula in the extremal limit (17)-(19) is quite good for the fundamental modes as long as  $r_1\kappa_1$  is small enough, while, as expected in [17], this approximation formula gets worse as the mode number is increased even for small values of  $r_1\kappa_1$ . This means that formula (17)-(19) does not give a correct asymptotic value of the QNM frequencies in the limit of large imaginary frequencies. Similar properties can be seen for other perturbations having different  $s$  and  $l$ . In Figure 2, we show the same results as those in Figure 1 but for the electromagnetic perturbations having  $l = 1$ . It is observed that similar behaviors of the QNM frequencies are seen in the case of electromagnetic perturbations, too.

Now, we explain our numerical results for the asymptotic behavior of the QNM of nearly extremal SdS spacetimes in the limit of large imaginary frequencies. In Figure 3, we plot the imaginary parts of the non-dimensional QNM frequencies,  $\omega/\kappa_1$ , of the gravitational perturbations with  $l = 2$  as a function of the mode number  $n$ . In this figure, the model parameter is  $r_1\kappa_1 = 10^{-3}$ . The figure shows that an asymptotic form of  $\text{Im}(\omega/\kappa_1) \sim n$ , which is similar to the analytical formula (17)-(19), is a good approximation for the imaginary parts in the limit of large imaginary frequencies. The same asymptotic form is inferred in all other QNM's we have calculated in the present study, regardless of the values of  $r_1\kappa_1$ ,  $l$ , and  $s$ .

Let us next focus on the behaviors of the real part of the QNM frequencies in the limit of highly damped modes. In Figures 4 and 5, the real parts of the non-dimensional mode frequencies,  $\omega/\kappa_1$  are plotted as a function of the imaginary parts of the frequencies up to sufficiently high-order modes for the  $l = 2, 3$  gravitational and for the  $l = 1, 2$  electromagnetic perturbations, respectively. The results for the model parameter of  $r_1\kappa_1 = 10^{-3}$  are shown in both figures. It is found that the real parts of the frequencies show oscillating behaviors as the imaginary parts of the frequencies are increased. It is important to note that similar oscillating behaviors have been observed in the QNMs of a Reissner-Nordström black hole. (The quasinormal fre-

quencies with large imaginary frequency of a Reissner-Nordström black hole can be given in terms of a solution of the algebraic equation [10, 12, 13],

$$e^{\beta\omega} + 2 + 3e^{k^2\beta\omega} = 0, \quad (20)$$

where  $\beta$  and  $k$  are contents determined with the mass and charge of the black hole (see, [10]). As shown by Neitzke [12] and Anderson and Howls [13], equation (20) has an infinite number of solutions and some solutions of equation (20) show periodicity.) The behavior of the amplitude of the oscillating  $\text{Re}(\omega/\kappa_1)$  as a function of  $\text{Im}(\omega/\kappa_1)$  resembles that of QNM frequencies of a Schwarzschild black hole. For the gravitational perturbations, the amplitude decreases for small values of  $\text{Im}(\omega/\kappa_1)$ , approaches the imaginary axis of the complex frequency plan, increases again, and finally approaches some constant value. The asymptotic value of the amplitude in the limit of highly damped modes seems to be non-zero constant, which is inferred as  $\sim 0.4$ . Therefore the limit of  $\text{Re}(\omega/\kappa_1)$  as  $\text{Im}(\omega/\kappa_1) \rightarrow \infty$  appears not to exist for the gravitational perturbations. For the electromagnetic perturbations, on the other hand, the amplitude of oscillating  $\text{Re}(\omega/\kappa_1)$  decreases monotonically as  $\text{Im}(\omega/\kappa_1)$  is increased. Its asymptotic value in the limit of highly damped modes seems to be zero. This means that for the electromagnetic perturbations, the limit of  $\text{Re}(\omega/\kappa_1)$  as  $\text{Im}(\omega/\kappa_1)$  goes to infinity seems to exist and to be zero. It is important to note that in a nearly extremal SdS black hole, the asymptotic behaviors of the quasinormal frequencies in the limit of highly damped modes are independent of the angular quantum number  $l$  of the perturbations. Although we do not show the results for other values of  $r_1\kappa_1$  and  $l$ , the asymptotic behavior of the QNM frequencies is not highly dependent on these parameters. In summary, our numerical results suggest that for the gravitational perturbations, the real parts of the QNM frequencies of nearly extremal SdS spacetimes do not go to any constant value in the limit of large imaginary frequencies because they show oscillating behaviors in the limit. For the electromagnetic perturbations, on the other hand, the real parts of the QNM frequencies seem to go to zero in the limit of large imaginary frequencies.

#### IV. CONCLUSIONS

We have calculated the high-order QNMs with large imaginary frequencies for the electromagnetic and gravitational perturbations in nearly extremal SdS spacetimes using Leaver's continued fraction method [20]. Our results show that for low-order QNMs, analytical formulas in the extremal limit derived by Cardoso and Lemos [16] and Maassen van den Brink [17] is a quite good approximation for the QNM frequencies as long as the model parameter  $r_1\kappa_1$  is small enough. For high-order QNMs, whose imaginary frequencies are sufficiently large, on the other hand, this formula becomes inaccurate even for

small values of  $r_1\kappa_1$ . Therefore, the approximation derived by Cardoso and Lemos cannot give correct asymptotic behaviors of the QNMs in the limit of large imaginary frequencies (see, also [17]). We also found that the real parts of the quasinormal frequencies have oscillating behaviors in the limit of highly damped modes. (Similar behaviors have been found in the quasinormal frequencies in a Reissner-Nordström black hole [10, 11, 12, 13].) The amplitude of oscillating  $\text{Re}(\omega)$  approaches a non-zero constant value for the gravitational perturbations and zero for the electromagnetic perturbations in the limit of highly damped modes, regardless of values of  $l$  and  $r_1\kappa_1$ . This means that for the gravitational perturbations, the real parts of the quasinormal frequencies of nearly extremal SdS spacetimes appear not to go to any constant value in the limit of highly damped modes. Therefore our numerical results suggest that Hod's conjecture is not applicable for nearly extremal SdS black holes because the limit of  $\text{Re}(\omega)$  as  $\text{Im}(\omega) \rightarrow \infty$  does not exist.

Although we computed high-order QNMs whose damping rates are quite large, all the QNMs we obtained in this study are still associated with a finite mode number but not infinity, because we investigated the properties of the QNMs with straightforward numerical approach. Thus, we cannot exclude the possibility that our numerical results do not show correct asymptotic behaviors. Other approaches to examine asymptotic behaviors of QNMs in the highly damping limit are necessary, in order to confirm our results of the asymptotic behaviors. As for high-order QNMs with large imaginary frequencies for moderate values of  $r_1\kappa_1$ , Leaver's method cannot be applied straightforwardly. Therefore, other numerical techniques are needed to obtain QNMs with large imaginary frequencies.

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### APPENDIX: SUPERPARTNER AND ISO-SPECTRAL RELATIONSHIP BETWEEN AXIAL AND POLAR PERTURBATIONS IN SDS SPACETIME

Non-radial gravitational perturbations of a SdS spacetime obey a Schrödinger-type wave equation, given by

$$\frac{d^2\phi^{(\pm)}}{dr_*^2} + [\omega^2 - V^{(\pm)}(r)]\phi^{(\pm)} = 0, \quad (\text{A.1})$$

where  $\phi^{(+)}$  and  $V^{(+)}$  ( $\phi^{(-)}$  and  $V^{(-)}$ ) are the gauge invariant perturbation function and the effective potential for polar (axial) parity perturbations, respectively. The effective potentials are given by

$$V^{(+)} = \frac{2\Delta}{r^5(cr + 3M)^2} \times [9M^3 + 9M^2cr + 3c^2Mr^2 + c^2(c+1)r^3 - 3M^2\Lambda r^3], \quad (\text{A.2})$$

$$V^{(-)} = \frac{2\Delta}{r^5} [(c+1)r - 6M], \quad (\text{A.3})$$

where  $c = (l+2)(l-1)/2$  (for detailed derivations of the master equation (A.1), see, e.g., [22, 24]). As shown first by Cardoso and Lemos [24], two potentials  $V^{(+)}$  and  $V^{(-)}$  are simply related through the relation, given by

$$V^{(\pm)} = \pm\beta \frac{df}{dr_*} + \beta^2 f^2 + \kappa f, \quad (\text{A.4})$$

where  $\beta = 6M$ ,  $\kappa = 4c(c+1)$ , and

$$f = \frac{\Delta}{2r^3(cr + 3M)}. \quad (\text{A.5})$$

This relation between two potentials are called the superpartner relationship. By virtue of the superpartner relationship, the perturbation function  $\phi^{(+)}$  ( $\phi^{(-)}$ ) can be written in term of  $\phi^{(-)}$  ( $\phi^{(+)}$ ) and its first derivative [29], given by

$$(\kappa \pm 2i\omega\beta)\phi^{(\pm)} = (\kappa + 2\beta^2 f)\phi^{(\mp)} \pm 2\beta \frac{d\phi^{(\mp)}}{dr_*}. \quad (\text{A.6})$$

It is worthwhile to note that the superpartner relationship (A.4) and (A.5) and the relations between two functions  $\phi^\pm$  (A.6) in a SdS spacetime have the same functional form as those in a Schwarzschild spacetime except for the definition of the function  $\Delta$  [29]. Since  $\Delta = \Lambda(r-r_1)(r_2-r)(r-r_3)r/3$ , the asymptotic form of the function  $f$  in the limit of  $r_* \rightarrow \pm\infty$  is given by

$$f \rightarrow \begin{cases} \frac{\Lambda(r_2-r_1)(r_1-r_3)}{6r_1^2(cr_1+3M)} e^{2|\kappa_1|r_*} & \text{as } r_* \rightarrow -\infty \\ \frac{\Lambda(r_2-r_1)(r_2-r_3)}{6r_2^2(cr_2+3M)} e^{-2|\kappa_2|r_*} & \text{as } r_* \rightarrow \infty. \end{cases} \quad (\text{A.7})$$

Then, it is easy to see that the function  $f$  has three properties; i) smooth for  $-\infty < r_* < \infty$ , ii)  $f$  and its derivatives of all orders vanish as  $r_* \rightarrow \pm\infty$ , iii) an integral  $\int_{-\infty}^{\infty} f dr_*$  exists. If the function  $f$  appearing in (A.4) satisfies three conditions above, as shown in [29], two potentials  $V^{(\pm)}$  give the same transmission amplitude and the same quasinormal frequencies. In a SdS spacetime, therefore, axial and polar perturbations yield the same set of quasinormal mode frequencies. This iso-spectral properties in a SdS spacetime is attributed to the fact that gravitational perturbations associated with a spin  $s = -2$  in a SdS spacetime can be described with a single Weyl scalar  $\Psi_4$  [23].

In a Schwarzschild-anti-de Sitter spacetime, exactly the same relations between polar and axial perturbations (A.4)–(A.6) obviously hold [24]. Yet, there is no iso-spectral property between polar and axial perturbations in a Schwarzschild-anti-de Sitter spacetime. In a Schwarzschild-anti-de Sitter spacetime, the master equation (A.1) does not have an asymptotic solution given by  $e^{\pm i\omega r_*}$  as  $r \rightarrow \infty$  and, furthermore,  $r_*$  has a finite range. The boundary condition at spatial infinity therefore must be modified. One of the plausible boundary

conditions is that perturbation functions vanish at spatial infinity, even though there are other options for the boundary conditions [24]. If this boundary condition is taken at spatial infinity, in general, the transformation (A.6) between polar and axial perturbations cannot hold this boundary condition. Therefore, the set of the quasinormal frequencies of polar perturbations in a Schwarzschild-anti-de Sitter spacetime is not the same as that of axial perturbations.

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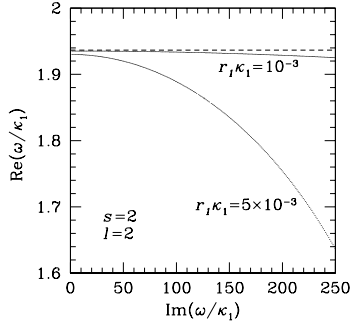


FIG. 1: Real parts of the non-dimensional QNM frequencies,  $\omega/\kappa_1$ , given as a function of the imaginary parts of the frequencies for the  $l=2$  gravitational perturbations. The model parameters  $r_1\kappa_1$  are taken to be  $r_1\kappa_1 = 10^{-3}$  and  $r_1\kappa_1 = 5 \times 10^{-3}$ . The frequencies obtained with the approximation formula in the limit of  $r_1\kappa_1 \rightarrow 0$  are also shown as the dashed curve.

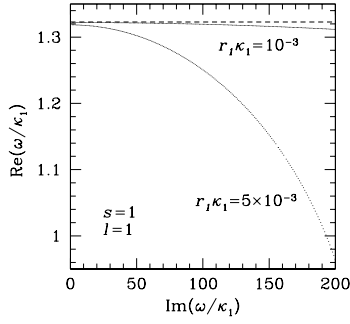


FIG. 2: Same as Figure 1 but for the  $l = 1$  electromagnetic perturbations.

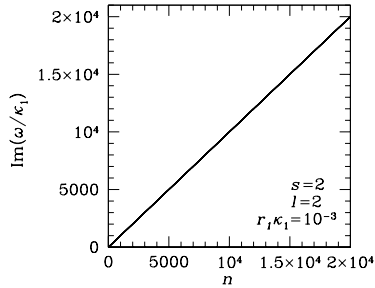


FIG. 3: Imaginary parts of the non-dimensional QNM frequencies,  $\omega/\kappa_1$ , given as a function of the mode number,  $n$ , for the gravitational perturbations associated with  $l = 2$ . The model parameter  $r_1\kappa_1$  is taken to be  $r_1\kappa_1 = 10^{-3}$ .

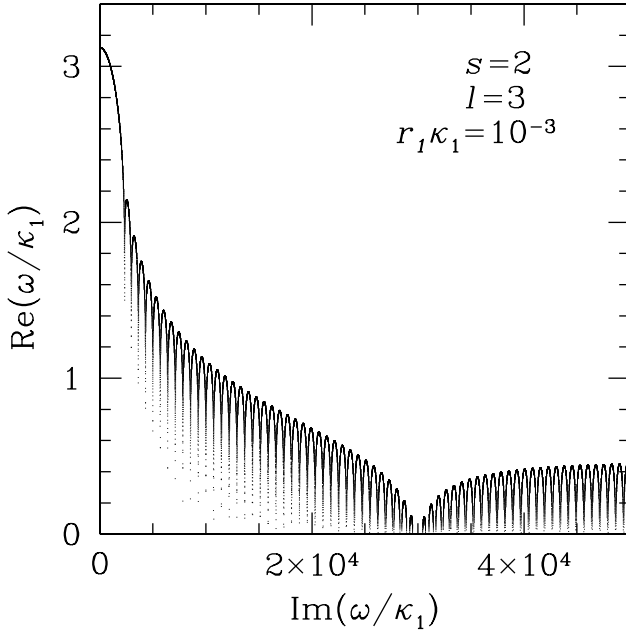
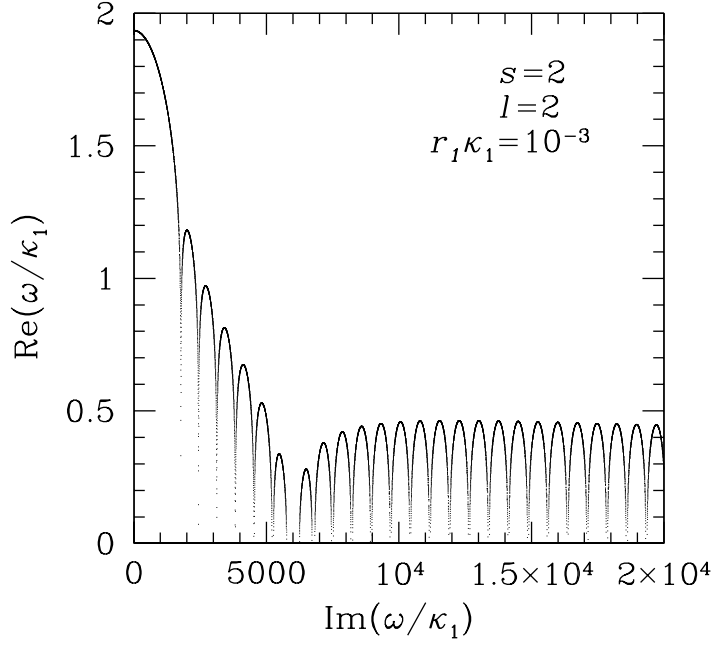


FIG. 4: Real parts of the non-dimensional QNM frequencies,  $\omega/\kappa_1$ , given as a function of the imaginary parts of the frequencies for the gravitational perturbations having  $l=2$  and  $l=3$ . The frequencies of the QNMs associated with different  $l$ ,  $l=2, 3$ , are shown in each panel. The model parameters  $r_1\kappa_1$  is taken to be  $r_1\kappa_1 = 10^{-3}$ .



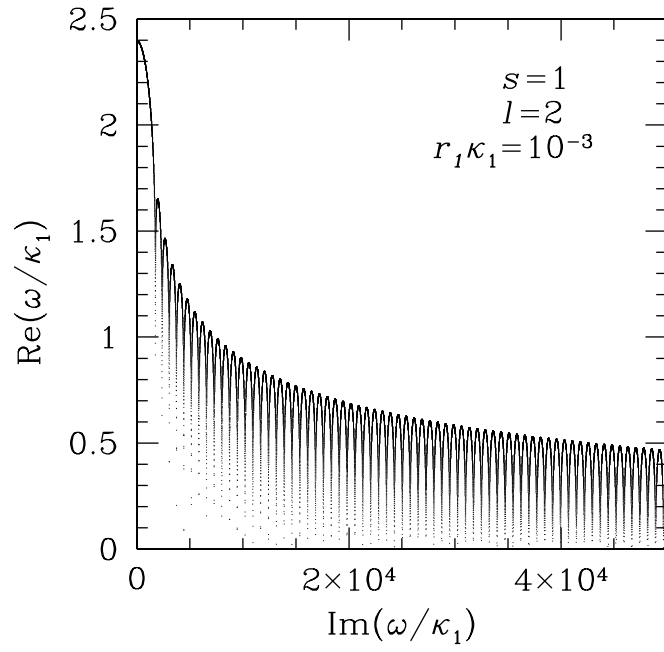
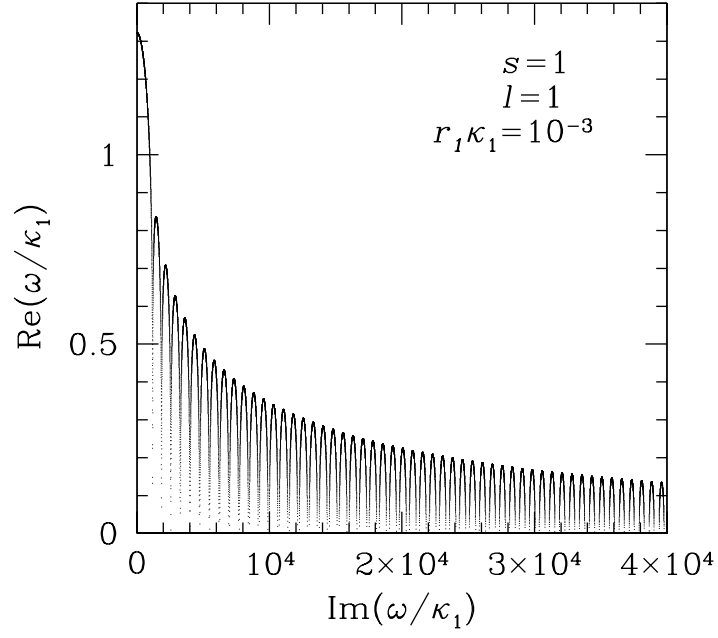


FIG. 5: Same as Figure 3 but for the electromagnetic perturbations having  $l = 1$  and  $l = 2$ .